METAMETRICS RESEARCH BRIEF

The Quantile® Framework for Mathematics Quantifies the Mathematics Ability Needed for College and Career Readiness

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July 2016

OBJECTIVE

The objective of this research is to answer the question, "What mathematics must a student be capable of performing to be ready for college or a career?" To address the question we analyzed mathematical concepts and skills that students may encounter as they begin their postsecondary education and/or enter the workplace. The answer is predicated on two perspectives: (a) mathematical readiness for college implies being ready for instruction in advanced mathematics courses associated with the beginning of the postsecondary educational experience; and, (b) readiness for the mathematical demands of careers implies, at a minimum, sufficient mathematical ability to perform well on the mathematics content required for a high school diploma. To answer the key question, we analyzed the difficulty of mathematical skills and concepts incorporated into the mathematics lessons found in mathematics texts commonly used in the United States. The Quantile® Framework for Mathematics provides the measurement foundation to place on a common scale both student mathematics ability and the difficulty of mathematical skills and concepts. Thus, we infer requisite student ability from the observed difficulty of mathematical skills and concepts contained in mathematics lessons presented in mathematics textbooks. We regard mathematics ability as an individual, malleable attribute, which improves with instruction and practice.

Key Hypotheses: Two key hypotheses are supported by the research findings:
1) The mathematics ability needed for college and career readiness ranges from approximately 1220Q to 1440Q.
2) The median mathematics demand for college and career readiness is 1350Q.

METHODS

Units of Analysis: At the most granular level, the Quantile Framework organizes mathematics content by mathematics skill and/or concept. To date, over 550 Quantile Skills and Concepts (QSCs) have been identified and calibrated to indicate their difficulty, or solvability, denominated in Quantile scale units. However, mathematics textbooks are often organized into lessons that focus on a small number of skills and concepts that relate to QSCs organized for instructional delivery. In a textbook, the entire spectrum of lessons represents the range of mathematics content demand placed on a student during a particular course in mathematics. Accordingly, the textbook lesson constitutes the unit of analysis for this study.

Sanford-Moore et al. (2014) presented a mathematics lesson continuum for K-12 mathematics. Their continuum culminated with the high school mathematics content associated with Algebra 1/Geometry/Algebra 2, or correspondingly, Integrated Mathematics 1, 2, and 3. The present research builds on their study by examining lessons associated with pre-calculus and trigonometry textbooks. In both studies, the units of analysis were mathematics lessons appearing in selected mathematics textbooks used in the public schools in the United States. Procedures for identifying and obtaining the pre-calculus and trigonometry textbooks to be analyzed for this study included a consideration of the market share in 2007 of mathematics textbooks. At that time, three major publishers controlled approximately 85% of the college textbook market and textbooks from these publishers were chosen for inclusion in the study. This percentage is consistent with the results from the 2013 Education Market Research report where 80.5% of the mathematics textbook market was dominated by three major publishers (EMR, 2014). Within each textbook, all lessons were calibrated to the Quantile Framework to produce a Quantile measure of difficulty.

In the present study, we augmented the lesson continuum of Sanford-Moore et al. (2014) by studying mathematics lessons from pre-calculus and trigonometry textbooks. We focused on pre-calculus and trigonometry skills and concepts because pre-calculus and trigonometry (a) constitute advanced topics of study beyond Algebra 2/Integrated Mathematics 3 and (b) are often the first courses encountered in postsecondary mathematics education for those students who did not take them in high school.

For purposes of this study, we treat pre-calculus and trigonometry as a sequel to the traditional high school mathematics sequence of Algebra 1/Geometry/Algebra 2 or correspondingly, Integrated Mathematics 1, 2, and 3. Thus, we regard the transition from those traditional high school pathways to pre-calculus and trigonometry as the frontier between high school mathematics and the mathematics of postsecondary education and career requirements. This perspective is supported by other sources. For example, as of 2016, nineteen states required Algebra 2 for graduation (Center for Public Education, 2013). The Common Core State Standards for Mathematics (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) also point to pathways that culminate with Algebra 2 or Integrated Mathematics 3. Furthermore, the National Mathematics Advisory Panel (2008) reported that "students who complete Algebra II are more than twice as likely to graduate from college compared to students with less mathematical preparation" (p. xiii).

Procedure:
We analyzed each mathematics textbook selected for the study and identified the lessons contained in the text. Typically, the publisher provided sufficient information to identify the lesson structure. Next, mathematics subject matter experts analyzed each lesson to identify the particular QSCs associated with that lesson. Then, the Quantile Framework provided the Quantile measure for each QSC. Finally, each pre-calculus or trigonometry lesson was assigned a single Quantile measure to represent the overall difficulty of the lesson (MetaMetrics, 2009a). The collection of lesson measures comprised the data for the study. However, two interpretive perspectives are necessary to accommodate

college and career readiness, as we explain next.

MetaMetrics created the Quantile Framework to facilitate classroom mathematics instruction. By design, the Quantile measure of a QSC corresponds to the mathematics ability a student needs to be ready for instruction on the QSC. The Quantile Framework predicts 50% understanding when an individual's mathematics ability equals the QSC difficulty. Thus, we say the default matching convention for the Quantile Framework is 50%. Anecdotal evidence collected from teachers (MetaMetrics, 2009b) suggests that the subjective experience of a student when the matching convention is 50% is that the student: (a) has the background knowledge needed to learn and apply the new information being taught; (b) can engage with the skills and concepts that are the focus of the instruction; (c) as a result of instruction, is able to solve problems utilizing those skills being taught; and, (d) appears to understand what he or she is learning. In essence, the 50% matching convention facilitates the interpretation that the student is ready for instruction. For purposes of this study, we use pre-calculus and trigonometry lesson measures assigned with a 50% matching convention to operationalize college readiness. On this basis, we infer the mathematics ability a student needs to be ready for instruction on mathematical content likely to be encountered in college.

When contemplating readiness for careers, however, a 50% understanding of mathematical skills and concepts seems insufficient, unless students are being recruited for an on-the-job instructional experience. The usual expectation in an occupational setting is that the individual has already been instructed on the relevant mathematical skills and concepts and consequently will be ready to perform in the job setting. A useful feature of the Quantile Framework is that its matching convention can be moderated to accommodate an interpretation of readiness to perform. Anecdotal evidence suggests that when the matching convention is 75%, the subjective experience of the student is such that the student: (a) is able to engage with the skills and concepts with minimal instruction, (b) is able to solve complex problems related to the skills and concepts, (c) is able to connect the skills and concepts with skills and concepts from other mathematics strands, and (d) experiences fluency and automaticity of skills. Because virtually all jobs require a high school diploma, and because many states require either Algebra 1 or Algebra 2 for a diploma, we consequently use high school mathematics lesson-measures from Algebra 1/Integrated Mathematics 1 through Algebra 2/Integrated Mathematics 3 and apply a 75% matching convention to represent the student ability needed for career readiness.

Measures:
The Quantile Framework uses a Rasch measurement model to provide developmentally-scaled measurements for students and mathematical skills or concepts. More specifically, the Quantile scale measures a student's mathematics ability as manifested through his or her mathematics achievement; and, it also measures the difficulty of mathematics skills and concepts. Importantly, students, skills and concepts are all measured using the same common scale, which allows teachers to provide mathematics instruction that is tailored to students' individual needs. MetaMetrics (2009a, 2014) provides additional information about the Quantile Framework and the Quantile scale.

ANALYSES

We analyzed pre-calculus and trigonometry lesson measures to determine (a) the median lesson difficulty and (b) how much variability in difficulty characterizes the pre-calculus and trigonometry lessons. We used SAS PROC UNIVARIATE to calculate descriptive summary statistics for the collection of lesson measures. To summarize the distribution, we report the 5th percentile, 25th percentile, median, 75th percentile, and 95th percentile of the distribution of the pooled pre-calculus and trigonometry lesson measures.

These results are graphically compared with the Grades 9-11 portion of the K-12 mathematics lesson continuum to depict the transition from high school to postsecondary mathematical content. Thus, for purposes of this study, pre-calculus and trigonometry lessons are associated with Grade 12.

RESULTS & DISCUSSION

The distributional summary statistics for pre-calculus and trigonometry lessons are displayed in Table 1 below. The median lesson measure is 1350Q. The interquartile range (IQR) extends from 1220Q to 1440Q representing the difficulty of the middle 50% of pre-calculus and trigonometry lessons. These findings correspond to the results in our two key hypotheses. It remains to justify these numbers as reasonable criteria for both college and career readiness.

Table 1
Selected Percentiles of the Distribution of Pre-calculus and Trigonometry Textbook Lesson Measures

Grade	Number of Lessons	5th	25th	50th	75th	95th
12	932	990Q	1220Q	1350Q	1440Q	1540Q

The justification for considering the values in Table 1 as an apt description of the mathematics ability needed for college readiness follows from two considerations. First, to the extent that pre-calculus and trigonometry represent the mathematical demand of topics taught near the beginning of college, they represent the mathematical demand encountered by students near the beginning of their postsecondary mathematics education. Second, the fact that the median and IQR represent the midpoint and the central 50% of the distribution of pre-calculus and trigonometry lesson difficulties implies that students with Quantile measures in this range are forecasted to be ready for instruction on typical postsecondary mathematics content. In essence, such students are ready for college insofar as their mathematical abilities are concerned.

To understand how the results in Table 1 can also serve as an apt description of career readiness, it is necessary to compare them with the high school mathematics lesson continuum. To facilitate that perspective, we refer to the data from Grades 9-11 of Sanford-Moore et al. (2014). Using that data to establish a reasonable characterization of career readiness, we first referred to the median lesson measures associated with the traditional and integrated high school mathematics pathways—Algebra 1/Integrated Mathematics 1; Geometry/Integrated Mathematics 2; and, Algebra 2/Integrated Mathematics 3. Next, using the Quantile Framework, we calculated the student ability needed to demonstrate 75% understanding of the median lesson in each course. The resulting Quantile measures represent career readiness because they correspond to 75% understanding of material typically associated with obtaining a high school diploma, which in turn is the minimum requirement for career entry in most instances. The two sets of numbers are displayed in Table 2 for comparison. It is readily apparent that the 75% matching convention applied to the traditional and integrated lesson medians yields nearly the same range as the 50% matching convention applied to the pre-calculus and trigonometry lessons.

Table 2

Comparison of College and Career Readiness Criteria

Career Readiness			College Readiness	
Content Area	**75% Understanding**		**50% Understanding**	**Pre-calculus & Trigonometry**
Algebra 2/Integrated Math 3	1418Q		1440Q	75th Percentile
Geometry/Integrated Math 2	1358Q		1350Q	Median
Algebra 1/Integrated Math 1	1218Q		1220Q	25th Percentile

The comparison summarized in Table 2 is rendered visually in Figure 1 in the context of the lesson continuum during the high school years. The box-and-whiskers plot for pre-calculus and trigonometry serves as the visual summary of lesson measures associated with college readiness. The IQR summarizes the typical range of mathematics demand and the median provides a one-number summary of the typical demand. In Figure 1, an arrow points to the IQR of the right-most box as the visual representation of college readiness. Similarly the triangles in the graph represent the Quantile measures that correspond to 75% understanding of median high school pathway subjects. To realize how a single result accommodates both college and career perspectives, consider the following question. What can we expect a student to be able to do if he or she has mathematics ability in the range 1220Q to1440Q? The answer is two-fold: (a) we expect the student to be ready for instruction in pre-calculus and trigonometry, and (b) we expect him or her to be able to perform with 75% understanding of the typical mathematics skills and concepts contained in traditional and integrated mathematics pathways. On this basis, we treat this range of student ability, 1220Q to 1440Q, as evidence that the student is ready for both college and career; and, we regard 1350Q as the mathematics ability at the midpoint of the college and career readiness range.

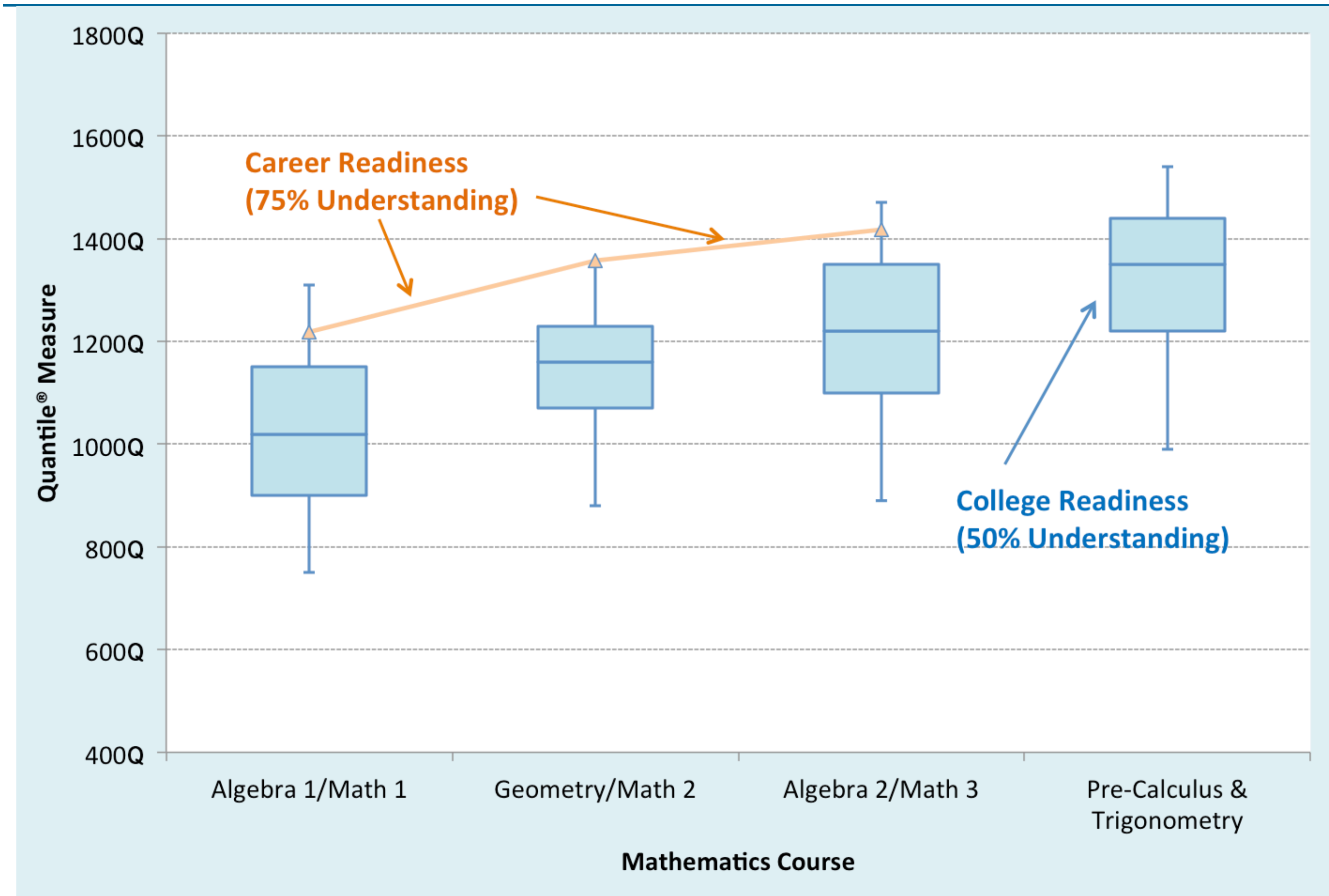

Figure 1. **Perspectives for college and career readiness.**

Based on these results we extend the Sanford-Moore et al. (2014) mathematics lesson continuum by including the results for pre-calculus and trigonometry lessons. The new updated continuum is presented in Figure 2.

Studies of mathematics textbook lessons using the Quantile Framework have provided a coherent internally consistent picture of the mathematics continuum for K-12, culminating with a proposed set of criteria for characterizing both college and career readiness. What sources of external evidence exist to support this characterization of CCR for mathematics? Currently, we can refer to three sources: (a) an analysis of the mathematics concepts and skills associated with a particular occupation (e.g., electrician); (b) a tentative link to the ACT benchmark for mathematics; and, (c) results from a state linking study, which characterizes the state's mathematics performance standard.

An analysis of the mathematics concepts and skills associated with the occupation of electrician (SOC 47-2111.00) lends tentative support for the college and career readiness findings. The mathematics concepts and skills needed to perform as an entry-level electrician were identified by Job Corps (2014). These concepts and skills were compared to the QSCs of the Quantile Framework and the student ability needed for 75% understanding was calculated to represent the mathematics demand for entry-level job performance. The median mathematics demand for electricians was 1045Q, and the IQR stretched from approximately 880Q to 1200Q. Thus, the typical mathematics demand for an entry-level electrician lies just below our proposed CCR range. This close proximity is reassuring although it does not provide definitive evidence for the proposed CCR range. Even so, a student pursuing a career as an electrician must realize that job accessibility involves not only being prepared for the performance demands of the job, but also satisfying the educational requirements for job entry. Electrician jobs most often require a high school diploma; on that basis, the occupation of electrician implicitly presumes student abilities in the range of 1220Q to 1440Q—the range of abilities needed to demonstrate 75% understanding of the typical mathematics lessons covered in courses required for high school graduation.

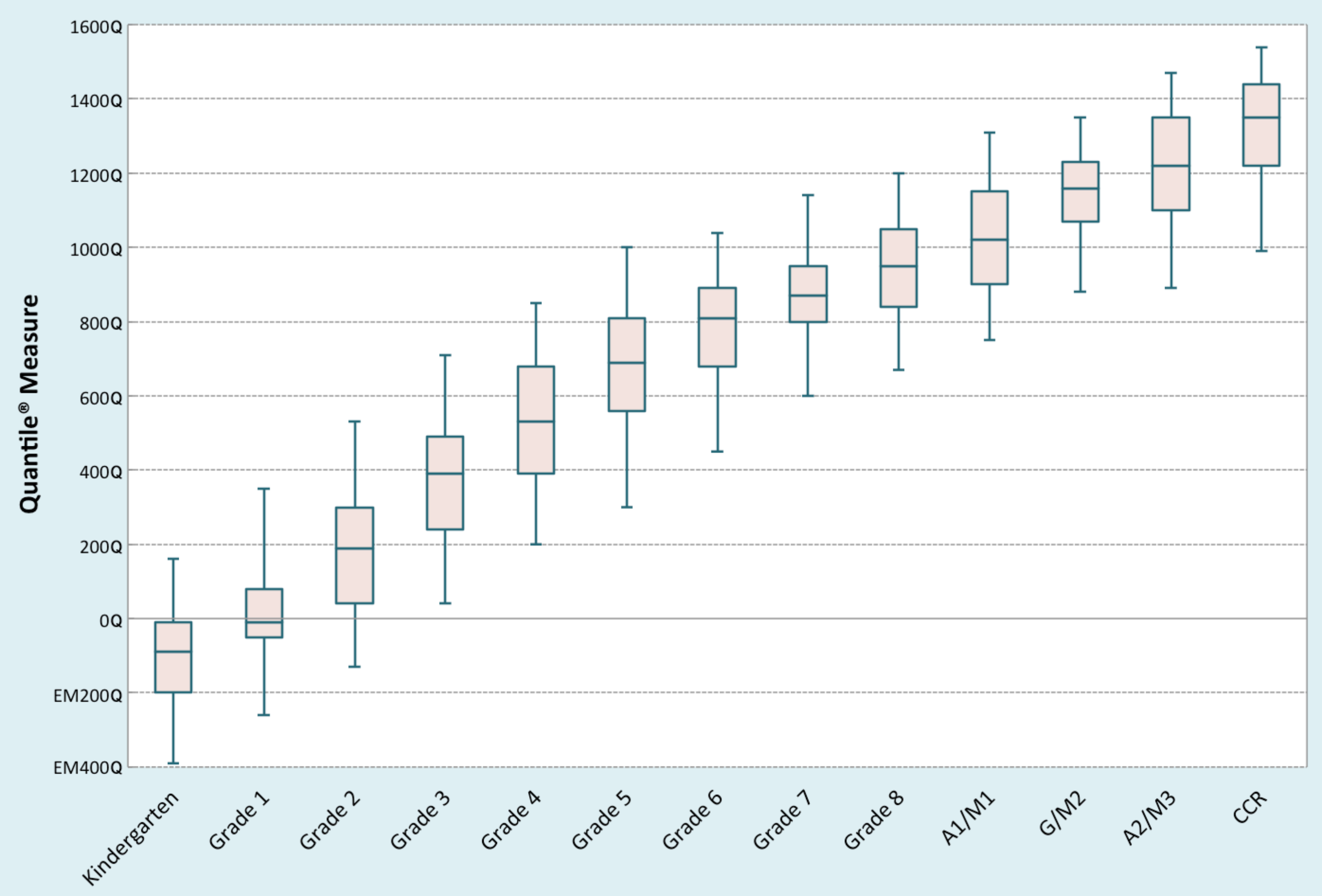

Figure 2. **K-12 Mathematics Lesson Continuum for College and Career Readiness (CCR)**

The ACT College Readiness Benchmark for Mathematics is 22 on the ACT mathematics scale. According to the College Board, "students who achieve this score on the ACT Mathematics Test have a 50% likelihood of achieving a B or better in a first-year College Algebra course at a typical college" (ACT, Inc., 2014, p. 1). The Quantile measure of the benchmark is 1200Q (MetaMetrics, 2013). Analogously, a linking study for one southern state identified 1235Q as the Quantile measure associated with the state's mathematics performance standard for Algebra II (MetaMetrics, 2012). These measures are very near the lower boundary of the IQR for mathematics CCR derived from our analyses.

As with any research study, the present study has several limitations. First and foremost, there was no definitive sampling frame for mathematics content. Our study results depend on (a) the mathematics texts admitted to the study, (b) the lessons identified in those texts, and (c) the mathematical skills and concepts that were the focus of the sampled lessons. To the extent that we failed to include some mathematics texts, lessons, skills or concepts, we could have produced biased estimates of the population percentiles that were focal results for the study. There is no way to know whether the final sample is sufficient. And, because it was not a probability sample, we cannot produce statistical estimates of the margin of error for our results. However, as explained in Sanford-Moore et al. (2014), we followed an objective strategy to identify systematically the content for the study.

The evidence for external validity presented in this research brief was limited. An important agenda for future study is the gathering of additional evidence to support our characterization of the mathematics ability needed for college and career readiness. For example, predictive evidence relating high school student mathematics abilities to initial scholastic performance or on-the-job performance would be one category of potential external validation. Another would be the charting of student growth and mathematical learning progressions across the boundary between K-12 and postsecondary realms of endeavor. Such studies might also yield other rationales for CCR mathematics standards—e.g., a rationale based on the alignment of student growth with mathematical demands, which change as students grow in their mathematics abilities. Another avenue for external validation is benchmarking the results reported here against national and international mathematics achievement when such results become available on the Quantile scale.

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